

# Approximate Deconvolution Model of Turbulence with Critical Regularization

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## Abstract

In this paper, we establish the existence of a unique “regular” weak solution to the Approximate Deconvolution Model (ADM) of turbulence with critical regularization.

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## 1 Introduction

### 1.1 Motivation

Let  $\mathbb{T}_3$  be the three dimensional torus  $\mathbb{T}_3 = (\mathcal{R}^3/\mathcal{T}_3)$  where  $\mathcal{T}_3 = 2\pi\mathbb{Z}^3/L$ ,  $L > 0$ ,  $0 \leq \theta \leq 1$ ,  $N \in \mathbb{N}$  and  $T \in (0, \infty)$ . We consider here the generalized Approximate Deconvolution Model (ADM) of order  $N$  introduced in [5],

$$(1.1) \quad \begin{cases} \frac{\partial \mathbf{w}}{\partial t} + \nabla(\overline{D_{N,\theta}(\mathbf{w}) \otimes D_{N,\theta}(\mathbf{w})}) - \nu \Delta \mathbf{w} + \nabla q = \overline{\mathbf{f}} & \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ \operatorname{div} \mathbf{w} = 0, \quad \int_{\mathbb{T}_3} \mathbf{w} = 0, \\ \mathbf{w}(t, \mathbf{x} + L\mathbf{e}_j) = \mathbf{w}(t, \mathbf{x}), \\ \mathbf{w}_{t=0} = \mathbf{w}_0 = \overline{\mathbf{v}_0}, \end{cases}$$

where  $D_{N,\theta}(\mathbf{w})$  is an generalized deconvolution operator given below (see section 1.3) and  $\overline{\mathbf{v}_0}$  is the initial condition of the mean Navier-Stokes Equations [5].

This family of equations approximates the Navier-Stokes equations [11] with periodic boundary conditions. When  $\theta = 1$  and for a fixed  $N > 0$ , we recover the van Cittert deconvolution operator used in Large eddy simulation (LES) by Stolz and Adams [1] see also [7]. When  $N = 0$  and  $\theta = 1$ , we obtain the LES model studied by Layton-Lewandowski in [9, 8, 10, 2] see also [6].

The authors in [5] showed the existence of a unique “regular” solution to the problem (1.1) with  $\theta > \frac{3}{4}$ . They conjecture also in [5] that this value  $\theta > \frac{3}{4}$  is not critical to ensure the existence and the uniqueness to the solution. Our task here is to find the critical regularization  $\theta$  (see Theorem 2.1) needed to establish global in time existence of a unique “regular” weak solution to problem (1.1).

Once existence and uniqueness of a “regular” weak solution to (1.1) with critical regularization is known, further theoretical properties of the model with critical and subcritical regularizations can then be developed. These are currently under study by the author, in particular the question of the limit as  $N \rightarrow \infty$  is work in progress, and will be presented in a subsequent report.

In order to make the paper self contained, after introducing relevant function spaces we recall from [5] how the generalized deconvolution operator  $D_{N,\theta}(\mathbf{w})$  is constructed. Then in section 2 we prove the existence of a unique “regular” solution to the problem (1.1) with  $\theta = \frac{1}{2}$ .

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## 1.2 Notations

In this section we fix notation of function spaces that we shall employ. We denote by  $L^p(\mathbb{T}_3)$  and  $H^s(\mathbb{T}_3)$ ,  $s \geq -1$ ,  $1 \leq p \leq \infty$ , the usual Lebesgue and Sobolev spaces over  $\mathbb{T}_3$ , and define the Bochner spaces  $C(0, T; X)$ ,  $L^p(0, T; X)$  in the standard way. The Sobolev spaces  $\mathbf{H}^s = H^s(\mathbb{T}_3)^3$ , of mean-free functions are classically characterized in terms of the Fourier series

$$\mathbf{H}^s = \left\{ \mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3} \mathbf{c}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, (\mathbf{c}_{\mathbf{k}})^* = \mathbf{c}_{-\mathbf{k}}, \mathbf{c}_0 = 0, \|\mathbf{v}\|_{s,2}^2 = \sum_{\mathbf{k} \in \mathcal{T}_3} |\mathbf{k}|^{2s} |\mathbf{c}_{\mathbf{k}}|^2 < \infty \right\},$$

where  $(\mathbf{c}_{\mathbf{k}}^N)^*$  denote the complex conjugate  $\mathbf{c}_{\mathbf{k}}^N$ . In addition we introduce

$$\begin{aligned} \mathbf{H}_{\text{div}}^s &= \{\mathbf{v} \in \mathbf{H}^s; \operatorname{div} \mathbf{v} = 0 \text{ in } \mathbb{T}_3\}, \\ \mathbf{H}^{-s} &= (\mathbf{H}^s)', \quad \mathbf{L}^2 = \mathbf{H}^0, \quad \mathbf{L}_{\text{div}}^2 = \mathbf{H}_{\text{div}}^0. \end{aligned}$$

Throughout we will use  $C$  to denote an arbitrary constant which may change from line to line.

## 1.3 The generalized deconvolution operator

Before formulating the main result of this paper, we recall how the generalized deconvolution operator is constructed [5, 3]. Let  $\alpha > 0$ ,  $s \geq -1$ ,  $0 \leq \theta \leq 1$ ,  $\mathbf{v} \in \mathbf{H}^s$  and let  $\bar{\mathbf{v}} \in \mathbf{H}^{s+2\theta}$  be the unique solution to the equations (compare to [13])

$$(1.2) \quad \alpha^{2\theta} (-\Delta)^\theta \bar{\mathbf{v}} + \bar{\mathbf{v}} = \mathbf{v},$$

$$(1.3) \quad \operatorname{div} \mathbf{v} = \operatorname{div} \bar{\mathbf{v}} = 0.$$

We also shall denote by  $\mathbb{A}_\theta$  the operator

$$(1.4) \quad \mathbb{A}_\theta : \begin{aligned} &\mathbf{H}^{s+2\theta} \longrightarrow \mathbf{H}^s, \\ &\mathbf{v} \longrightarrow \alpha^{2\theta} (-\Delta)^\theta \mathbf{v} + \mathbf{v}. \end{aligned}$$

The non-local operator  $\mathbb{A}_\theta$  is defined through the Fourier transform

$$(1.5) \quad \widehat{\mathbb{A}_\theta \mathbf{v}}(\mathbf{k}) = \left(1 + \alpha^{2\theta} |\mathbf{k}|^{2\theta}\right) \widehat{\mathbf{v}}(\mathbf{k}).$$

Therefore, one has

$$(1.6) \quad \bar{\mathbf{v}} = \mathbb{A}_\theta^{-1} \mathbf{v}.$$

Let us consider the operators

$$D_{N,\theta} = \sum_{i=0}^N (I - \mathbb{A}_\theta^{-1})^i.$$

A straightforward calculation yields

$$(1.7) \quad D_{N,\theta} \left( \sum_{\mathbf{k} \in \mathcal{I}_3} \mathbf{c}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \right) = \sum_{\mathbf{k} \in \mathcal{I}_3} \left(1 + \alpha^{2\theta} |\mathbf{k}|^{2\theta}\right) \left(1 - \left( \frac{\alpha^{2\theta} |\mathbf{k}|^{2\theta}}{1 + \alpha^{2\theta} |\mathbf{k}|^{2\theta}} \right)^{N+1}\right) \mathbf{c}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}.$$

Thus

$$(1.8) \quad D_{N,\theta} \left( \sum_{\mathbf{k} \in \mathcal{I}_3} \mathbf{c}_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{x}} \right) = \sum_{\mathbf{k} \in \mathcal{I}_3} \widehat{D_{N,\theta}}(\mathbf{k}) \mathbf{c}_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{x}},$$

where we have for all  $\mathbf{k} \in \mathcal{I}_3$ ,

$$(1.9) \quad 1 \leq \widehat{D_{N,\theta}}(\mathbf{k}) \leq N + 1 \quad \text{for each } N > 0$$

$$(1.10) \quad \text{and } \widehat{D_{N,\theta}}(\mathbf{k}) \leq \widehat{\mathbb{A}}_{\theta} := \left( 1 + \alpha^{2\theta} |\mathbf{k}|^{2\theta} \right) \text{ for a fixed } \alpha > 0.$$

One can prove the following Lemma (see in [5]):

**Lemma 1.1** *For all  $s \geq -1$ ,  $\mathbf{k} \in \mathcal{I}_3$ , and for each  $N > 0$  there exist a constant  $C > 0$  such that for all  $\mathbf{v}$  sufficiently smooth we have*

$$(1.11) \quad \|\mathbf{v}\|_{s,2} \leq \|D_{N,\theta}(\mathbf{v})\|_{s,2} \leq (N + 1) \|\mathbf{v}\|_{s,2},$$

$$(1.12) \quad \|\mathbf{v}\|_{s,2} \leq C \|D_{N,\theta}(\mathbf{v})\|_{s,2} \leq C \|\mathbb{A}_{\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}}(\mathbf{v})\|_{s,2},$$

$$(1.13) \quad \|\mathbb{A}_{\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}}(\overline{\mathbf{v}})\|_{s,2} \leq \|\mathbf{v}\|_{s,2},$$

$$(1.14) \quad \|\mathbf{v}\|_{\theta+s,2} \leq C(\alpha) \|\mathbb{A}_{\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}}(\mathbf{v})\|_{s,2}.$$

## 2 Main result: Existence and uniqueness in the critical case

We present our main result, restricting ourselves to the critical case  $\theta = \frac{1}{2}$ , and for simplicity we drop some indices of  $\theta$ . Thus we will write “ $D_N$ ” instead of “ $D_{N,\theta}$ ” and “ $\mathbb{A}$ ” instead of “ $\mathbb{A}_{\theta}$ ” expecting that no confusion will occur.

**Theorem 2.1** *Assume that  $\theta = \frac{1}{2}$ . Let  $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1})$  be a divergence free function and  $\mathbf{v}_0 \in L^2_{\text{div}}$ . Then there exist  $(\mathbf{w}, q)$  a unique “regular” weak solution to problem (1.1) such that*

$$(2.1) \quad \mathbf{w} \in \mathcal{C}(0, T; \mathbf{H}^{\frac{1}{2}}_{\text{div}}) \cap L^2(0, T; \mathbf{H}^{1+\frac{1}{2}}_{\text{div}}),$$

$$(2.2) \quad \mathbf{w}_{,t} \in L^2(0, T; \mathbf{H}^{-\frac{1}{2}}),$$

$$(2.3) \quad q \in L^2(0, T; H^{\frac{1}{2}}(\mathbb{T}_3)),$$

fulfill

$$(2.4) \quad \begin{aligned} \int_0^T \langle \mathbf{w}_{,t}, \boldsymbol{\varphi} \rangle - \overline{(D_N(\mathbf{w}) \otimes D_N(\mathbf{w}), \nabla \boldsymbol{\varphi})} + \nu(\nabla \mathbf{w}, \nabla \boldsymbol{\varphi}) - (q, \text{div } \boldsymbol{\varphi}) dt \\ = \int_0^T \langle \overline{\mathbf{f}}, \boldsymbol{\varphi} \rangle dt \quad \text{for all } \boldsymbol{\varphi} \in L^2(0, T; \mathbf{H}^{\frac{1}{2}}). \end{aligned}$$

Moreover,

$$(2.5) \quad \mathbf{w}(0) = \mathbf{w}_0.$$

**Proof of Theorem 2.1.**

The proof of Theorem 2.1 follows from the Galerkin method. This is a classical argument which we will not repeat here. For further information, we refer the reader to [14, 5, 4] and the references therein. We shall focus our attention on the a priori estimates and the compactness properties of problem (1.1) and finally we show that the solution we constructed is unique thanks to Gronwall's lemma.

**Step 1** (A priori estimates)

Multiplying (1.1) with  $\mathbb{A}D_N(\mathbf{w})$ , integrating over time from 0 to  $T$  and using the following identities

$$(2.6) \quad (\mathbf{w}_{,t}, \mathbb{A}D_N(\mathbf{w})) = \frac{1}{2} \frac{d}{dt} \|\mathbb{A}^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})\|_2^2$$

$$(2.7) \quad (-\Delta \mathbf{w}, \mathbb{A}D_N(\mathbf{w})) = \|\mathbb{A}^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})\|_{1,2}^2,$$

$$(2.8) \quad \langle \bar{\mathbf{f}}, \mathbb{A}D_N(\mathbf{w}) \rangle = \langle \mathbb{A}^{\frac{1}{2}} D_N^{\frac{1}{2}}(\bar{\mathbf{f}}), \mathbb{A}^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}) \rangle,$$

and

$$(2.9) \quad \begin{aligned} \left( \overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})}, \nabla \mathbb{A}D_N(\mathbf{w}) \right) &= (D_N(\mathbf{w}) \otimes D_N(\mathbf{w}), \nabla D_N(\mathbf{w})) \\ &= - \left( \operatorname{div} D_N(\mathbf{w}), \frac{|D_N(\mathbf{w})|^2}{2} \right) = 0, \end{aligned}$$

leads to the a priori estimate

$$(2.10) \quad \sup_{t \in [0, T]} \|\mathbb{A}^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})\|_2^2 + \nu \int_0^t \|\mathbb{A}^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})\|_{1,2}^2 ds \leq C(\mathbf{v}_0, \mathbf{f}),$$

see in [5] for the details.

We deduce from (2.10) and Lemma 1.1 that

$$(2.11) \quad D_N(\mathbf{w}) \in L^\infty(0, T; \mathbf{L}_{\operatorname{div}}^2) \cap L^2(0, T; \mathbf{H}^1(\mathbb{T}_3)^3),$$

and

$$(2.12) \quad \mathbf{w} \in L^\infty(0, T; \mathbf{H}_{\operatorname{div}}^{\frac{1}{2}}) \cap L^2(0, T; \mathbf{H}^{1+\frac{1}{2}}(\mathbb{T}_3)^3).$$

We observe from (2.11) that

$$(2.13) \quad D_N(\mathbf{w}) \otimes D_N(\mathbf{w}) \in L^2(0, T; H^{-\frac{1}{2}}(\mathbb{T}_3)^{3 \times 3}).$$

Thus from (2.13) and (1.2) we obtain

$$(2.14) \quad \overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} \in L^2(0, T; H^{\frac{1}{2}}(\mathbb{T}_3)^{3 \times 3}).$$

For the pressure term  $q$ , we deduce from (1.1) that it verifies the following equation

$$(2.15) \quad \Delta q = - \operatorname{div} \operatorname{div} \left( \overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} \right).$$

consequently the classical elliptic theory combined with (2.14) implies that

$$(2.16) \quad \int_0^T \|q\|_{\frac{1}{2}, 2}^2 dt < K.$$

From eqs. (1.1) and (2.14) we also obtain that

$$(2.17) \quad \int_0^T \|\mathbf{w}_{,t}\|_{-\frac{1}{2},2}^2 dt < K.$$

**Step 2** (Passing to the limit in the equations)

Now consider a sequence  $(\mathbf{w}^n, p^n)$  of approximated solutions to (1.1) (This sequence can be constructed via Galerkin method [14, 5, 4]). We have to prove that this sequence converges, up to a subsequence, to a solution to problem (1.1). From the above established a priori estimates we can deduce that this sequence verifies

$$(2.18) \quad \mathbf{w}^n \in L^\infty(0, T; \mathbf{H}_{\text{div}}^{\frac{1}{2}}) \cap L^2(0, T; \mathbf{H}^{1+\frac{1}{2}}),$$

$$(2.19) \quad q^n \in L^2(0, T; H^{\frac{1}{2}}(\mathbb{T}_3)),$$

$$(2.20) \quad \mathbf{w}_{,t}^n \in L^2(0, T; \mathbf{H}^{-\frac{1}{2}}).$$

The above established a priori estimate combined with Aubin-Lions compactness Lemma [12] are clearly sufficient for concluding that the limit as  $n \rightarrow \infty$ , the solution  $(\mathbf{w}, q)$  satisfies (2.4). Moreover, from (2.18) and (2.20) one can deduce by a classical argument (see [3]) that

$$(2.21) \quad \mathbf{w} \in \mathcal{C}(0, T; \mathbf{H}^{\frac{1}{2}}).$$

Furthermore, from the strong continuity of  $\mathbf{w}$  with respect to time and with value in  $\mathbf{H}^{\frac{1}{2}}$  we deduce that  $\mathbf{w}(0) = \mathbf{w}_0$ .

Let us mention also that  $\mathbb{A}D_N(\mathbf{w})$  is a possible test function in the weak formulation (2.4).

Thus  $\mathbb{A}^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{w})$  verifies for all  $t \in [0, T]$  the following equality

$$(2.22) \quad \begin{aligned} & \left( \|\mathbb{A}^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{w})(t)\|_2^2 \right) + 2\nu \int_0^t \left( \|\mathbb{A}^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{w})\|_{1,2}^2 \right) ds \\ &= 2 \int_0^t \langle \mathbf{f}, D_N(\mathbf{w}) \rangle ds + \|\mathbb{A}^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{w}_0)\|_2^2. \end{aligned}$$

**Step 3** (Uniqueness)

Since the pressure part of the solution is uniquely determined by the velocity part it remain to show the uniqueness to the velocity.

Next, we will show the continuous dependence of the solutions on the initial data and in particular the uniqueness.

Let  $(\mathbf{w}_1, q_1)$  and  $(\mathbf{w}_2, q_2)$  be any two solutions of (1.1) on the interval  $[0, T]$ , with initial values  $\mathbf{w}_1(0)$  and  $\mathbf{w}_2(0)$ . Let us denote by  $\delta\mathbf{w} = \mathbf{w}_2 - \mathbf{w}_1$ . We subtract the equation for  $\mathbf{w}_1$  from the equation for  $\mathbf{w}_2$  and test it with  $\mathbb{A}D_N(\delta\mathbf{w})$ . We get using successively the fact that the averaging operator commutes with differentiation under periodic boundary conditions, the norm duality, Young inequality, Lemma 1.1, Hölder inequality and Sobolev embedding theorem:

$$(2.23) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbb{A}^{\frac{1}{2}}D_N^{\frac{1}{2}}\delta\mathbf{w}\|_{\frac{1}{2},2}^2 + \nu \|\nabla \mathbb{A}^{\frac{1}{2}}D_N^{\frac{1}{2}}\delta\mathbf{w}\|_2^2 \\ & \leq \overline{(D_N(\mathbf{w}_2) \otimes D_N(\mathbf{w}_2) - D_N(\mathbf{w}_1) \otimes D_N(\mathbf{w}_1), \nabla \mathbb{A}D_N(\delta\mathbf{w}))} \\ & \leq (D_N\mathbf{w}_2 \cdot \nabla D_N\mathbf{w}_2 - D_N\mathbf{w}_1 \cdot \nabla D_N\mathbf{w}_1, D_N\delta\mathbf{w}) \\ & \leq \frac{4}{\nu} \|D_N\delta\mathbf{w} \cdot \nabla D_N\mathbf{w}_1\|_{-1,2}^2 \\ & \leq \frac{4(N+1)^2}{\nu} \|\mathbb{A}^{\frac{1}{2}}D_N^{\frac{1}{2}}\delta\mathbf{w}\|_2^2 \|\nabla \mathbf{w}_1\|_{\frac{1}{2}}^2. \end{aligned}$$

Using Gronwall's inequality we conclude the continuous dependence of the solutions on the initial data in the  $L^\infty([0, T], \mathbf{H}^{\frac{1}{2}})$  norm. In particular, if  $\delta \mathbf{w}_0 = 0$  then  $\delta \mathbf{w} = 0$  and the solutions are unique for all  $t \in [0, T]$ . Since  $T > 0$  is arbitrary this solution may be uniquely extended for all time.

This finishes the proof of Theorem 2.1.

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